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A Method for Solving $J_1(x)\, Y_1(\rho x) - J_1(\rho x)\, Y_1(x) = 0$

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INTRODUCTION

The equation

$$J_1(x)\, Y_1(\rho x) - J_1(\rho x)\, Y_1(x) = 0 \tag{1}$$

is of importance in the analytical solution of Reynold's equation [1] for a tilted rectangular pad-slider bearing [2, 3]. The parameter ρ (> 1) is the ratio of the inlet height to outlet height of the bearing. Muskat *et al.* [2] emphasize that in order to obtain reliable results for various parameters of the bearing, such as its load capacity, it is necessary to calculate x_n ($n = 1, 2,...$) the roots of Eq. (1) to high accuracy, i.e., to better than 7 decimal places for $\rho = 10$ and for low values of n. These authors used graphical interpolation and tables of J_1 and Y_1 [4, 5] to calculate x_n.

In principle, published tables of $x_n(\rho)$ (see, e.g., [2, 6–11]) could be stored in a computer and interpolation could be used to obtain x_n for any given ρ but it would be better to generate x_n directly, without reference to stored values, many of which would probably not be needed in the interpolation.

The solution of Eq. (1) described in this article has the following advantages:

(1) It is simple and renders unnecessary, tedious calculation of interpolated values from published high-accuracy tables.

(2) It uses the amplitude $M_1(x)$ and phase $\theta_1(x)$ of first-order Bessel functions, thus converting the problem from finding the zeros of an oscillating function into the problem of finding the zero of the function $\theta_1(\rho x) - \theta_1(x) - n\pi$ which has only one zero for a given n and ρ.

(3) It uses the quadratically convergent-Newton's method which is ideally suited to finding the zero of the new nonoscillating function but which is not well suited to finding the zeros of the original oscillatory function of Eq. (1).

(4) The method, because of its rapid convergence and relatively small requirements for programme storage space is suited to small desk-top calculators or even programmable hand calculators.

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Theory

When the Bessel functions $J_1(x)$ and $Y_1(x)$ are written in terms of their modulus $M_1(x)$ and their phase $\theta_1(x)$, Eq. (1) becomes

$$M_1(x)\, M_1(\rho x) \sin\{\theta_1(\rho x) - \theta_1(x)\} = 0, \tag{2}$$

whose solution for x- and ρ-finite is

$$\theta_1(\rho x_n) - \theta_1(x_n) = n\pi, \qquad n = 1, 2, 3, \ldots . \tag{3}$$

Equation (3) and, hence, Eq. (1) may be solved by using the Newton iteration [9, Sec. 3.9.5],

$$x_{n,m+1} = x_{n,m} - (\theta_1(x_{n,m}) - \theta_1(x_{n,m}) - n\pi)/(\theta_1'(\rho x_{n,m}) - \theta_1'(x_{n,m})), \tag{4}$$

where $x_{n,m}$ is the mth approximation to x_n. Use of [9, Eq. 9.2.21] for $\theta_1'(x)$ gives

$$x_{n,m+1} = x_{n,m} - \frac{\{\theta_1(\rho x_{n,m}) - \theta_1(x_{n,m}) - n\pi\}}{(2/\pi x_{n,m})\{(1/M_1^2(\rho x_{n,m})) - (1/M_1^2(x_{n,m}))\}}. \tag{5}$$

This simplification is particularly desirable because it means that only amplitudes and phases are required for the final iteration (Eq. (5)). If (1) were to be solved using Newton's iteration, then it would be necessary to calculate the four functions J_1, J_1', Y_1, Y_1' instead of the two required for the present method. Of the various approximations available for Bessel functions [9, 12, 13] those of [9, Secs. 9.4.4–9.4.6] were used in the present work because they give $M_1(x)$ and $\theta_1(x)$ explicitly for $x \geqslant 3$. When $x < 3$ approximations for $J_1(x)$ and $Y_1(x)$ are given and these are easily converted to $M_1(x)$ and $\theta_1(x)$ using standard techniques.

A suitable starting value $x_{n,1}$ for iteration (5) is

$$x_{n,1} = n\pi/(\rho - 1) \tag{6}$$

which, in fact, is the first term of McMahon's approximation for x_n ([9, Secs. 9.5.28 and 9.5.29; 14]).

Error analysis based on simple calculus expansions shows that the error ε_A introduced into x_n by the use of the approximations for $M_1(x)$ and $\theta_1(x)$ [9] is

$$\varepsilon_A \leqslant 6 \times 10^{-8} x_n/n. \tag{7}$$

This accuracy is sufficient for calculations on bearings. In those cases where the magnitude of the fourth term of the McMahon's series expansion for x_n (see [9, Secs. 9.5.28 and 9.5.29]) was less than the ε_A of Eq. (7), the McMahon solution was of course used because of its smaller error.

In order to decide at what point the Newton iteration should be truncated it is necessary to determine the minimum value of ε_A for a required range of ρ. The substitution of $x_{n,1}$ from Eq. (6) for x_n in Eq. (7) shows that ε_A is of order of $6\pi \times 10^{-8}/(\rho - 1)$, thus, in the range $1 < \rho \leqslant 100$, which extends well beyond that commonly used for practical bearings, the smallest value of ε_A is

$$(\varepsilon_A)_{\min} = 1.9 \times 10^{-9}, \qquad 1 < \rho \leqslant 100. \tag{8}$$

When the notation of Scarborough [15] is adapted to the present article it follows that

$$\varepsilon_N = |x_{n,m+1} - x_n| \leqslant |(\mu/2f'(x_{n,m}))(x_{n,m+1} - x_{n,m})^2|, \tag{9}$$

where ε_N is the error due to truncation of Newton's iteration at $x_{n,m+1}$ and μ is the maximum value of $f''(x)$ in the interval $x_{n,m}$ to x_n. For the present purpose it is sufficient to set $\mu = f''(x_{n,m+1})$. With this substitution it is found that

$$\varepsilon_N = |x_{n,m+1} - x_{n,m}| \leqslant 0.58(x_{n,m+1} - x_{n,m})^2, \qquad 1 < \rho \leqslant 100. \tag{10}$$

The minimum error given in Eq. (8) is set by the approximations used for $M_1(x)$ and $\theta_1(x)$, but ε_N can be made as small as required (subject, of course, to rounding errors) simply by increasing m. It was decided to stop iteration (5) when

$$|x_{n,m+1} - x_{n,m}| \leqslant 10^{-5}, \tag{11}$$

thus ensuring from (10) that

$$\varepsilon_N < 5.8 \times 10^{-11}, \qquad 1 < \rho \leqslant 100, \tag{12}$$

i.e., $\varepsilon_N \ll (\varepsilon_A)_{\min}$ (see Eq. (8)). For those roots which required the use of Newton's iteration it was found that truncation criterion (11) was fulfilled by the vast majority of the roots after two iterations. In some cases three iterations were needed to determine the first root x_1 to the required accuracy.

In the slider-bearing problem, ρ is greater than unity, however, tables of x_n exist for $0 < \rho < 1$ [7, 8, 10, 11], thus it is of interest to relate the present work to the case $\rho < 1$. This is easily done by the transformation

$$\eta = 1/\rho; \qquad y = \rho x \tag{13}$$

in which case (1) becomes

$$J_1(\eta y)\,Y_1(y) - J_1(y)\,Y_1(\eta y) = 0. \tag{14}$$

The solutions x_n $(\rho > 1)$ of (1) can easily be transformed to the solutions y_n $(\eta < 1)$ of (14) simply by using (13). When this was done so as to obtain y_n in the range $0.01 < \eta < 0.99$ it was found that the results agreed to 7 decimals or better with the 10-decimal results of Fettis and Caslin [10, 11] in accord with the error ε_A of Eq. (7).

Concluding Remarks

The method of determining x_n described in this article is significantly faster than competitive methods such as the method of false position. The values of x_n are sufficiently accurate (7 decimals) for calculations on bearings and these calculations are further facilitated by the fact that tables of $x_n(\rho)$ and tedious interpolations are not required. The method is so successful that it can even be used on a Hewlett-Packard HP65 hand calculator, each iteration requiring the use of only two double-sided magnetic cards.

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